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Tensor Robust Principal Component Analysis with Continuum Modeling of Traffic Flow: Application to Abnormal Traffic Pattern Extraction in Large Transportation Networks

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Abstract

The study addresses the needs of detection and description of abnormal traffic patterns in large transportation networks formed due to the presence of unexpected disruptions, such as natural or manmade disasters. In order to take into account complex spatiotemporal structure of traffic dynamics and preserve multi-mode correlations, tensor-based traffic data representation is put forward. Further, with the reasonable assumptions on normal or expected traffic dynamics to exhibit similar periodic structure, the problem of abnormal or unexpected traffic patterns detection is treated as a low-rank modeling problem. More precisely, tensor robust principal component analysis is applied for the purpose of discovering distinctive normal and abnormal traffic patterns. For the validation purposes, continuum modeling approach is employed to emulate traffic dynamics, with consideration of the effect of aforementioned disruptions. The results suggested the applicability of proposed approach in order to extract abnormal traffic patterns in large transportation networks.

Keywords: Tensor Robust Principal Component Analysis; Anomaly Detection; Continuum Modeling Approach

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1. Introduction

Comprehensive analysis and proper description of traffic dynamics in large transportation networks are indeed a complicated procedures, yet necessary for many areas of transportation research. The situation is becoming even more challenging, considering the existence of unexpected disruptions, such as natural or manmade disasters. These disasters have a disruptive effect on traffic conditions and typically result in formation of abnormal traffic patterns. Among different methods proposed to analyze and extract meaningful information about aforementioned abnormal traffic patterns, a considerable amount utilizes statistical techniques with either temporal or spatial information regarding traffic flows. However, the traffic dynamics in large transportation networks usually exhibit complex spatiotemporal structure, which is one of the key obstacles on the way to their analysis. Therefore, spatiotemporal dependencies have to be modeled carefully and be considered simultaneously. This aspect was highlighted in lesser number of studies, for instance in Rempe et al. (2016) and Li et al. (2013), yet claimed to be an essential component for accurate description of traffic patterns.

The issue of accurate modeling of complex spatiotemporal dependencies is closely related to the mathematical tools used for analysis. For instance, methods relying on matrix-based traffic data representation are not capable of handling dependencies along more than two modes of traffic data simultaneously. Therefore, the dependencies, for example, only along one spatial and one temporal mode (Goulart et al., 2017) could be simultaneously taken into account. In order to overcome this issue and take into account dependencies along larger number of modes, other mathematical tools than matrices have to be used.

One of the possible solutions is utilization of tensor-based techniques. Tensors, being a higher order generalization of such mathematical objects as scalars, vectors and matrices were recently introduced to transportation domain. Tensor-based methods operate with multi-way matrices in order to capture underlying multi-mode structure of traffic dynamics (Ran et al., 2016). This feature of tensors, namely the ability to capture and preserve multi-mode correlations appeared to be important for such applications as imputation of missing traffic data obtained from sensors (Ran et al., 2016; Chen et al., 2018) and analysis of traffic dynamics in large-scale urban areas (Han et al., 2016).

In current study, the problem of abnormal traffic pattern extraction in large transportation networks is considered. In order to take into account complex spatiotemporal structure of traffic dynamics, aforementioned tensor-based traffic data representation is adopted first. Further, with the reasonable assumptions that traffic conditions and therefore normal traffic patterns have spatiotemporal correlations and demonstrate similar periodic structure (i.e. similar day-to-day dynamics inside a particular urban region), the problem of abnormal traffic pattern extraction is solved with the help of tensor robust principal component analysis. More precisely, it is assumed that the real-world observations of traffic dynamics could be decomposed into so-called low-rank and sparse components. The low-rank component contains the information about expected or normal traffic patterns. On the other hand, the sparse component depicts unexpected or abnormal traffic patterns. This decomposition into low-rank and sparse components is formulated as an optimization problem first, and further solved with the help of Augmented Lagrangian Method. This approach is capable to separate normal and abnormal traffic patterns in a robust way, taking into account the possibility of existence of grossly corrupted observations (non-Gaussian noise) in traffic data.

In order to validate aforementioned tensor-based abnormal traffic pattern extraction approach, the simulation of traffic dynamics under different normal and abnormal conditions had been conducted. More precisely, in current study the continuum modeling approach, extensively studied by Du et al. (2013), Xia and Wong (2009), Jiang et al. (2009) and others had been adapted. According to this continuum modeling approach, real road network with complex topology is viewed as a continuum, with travelers able to travel freely in two dimensional space. This transition from the discrete representation, where each road link in a network is a subject for a separate analysis, allow to easier the problem connected with the presence of enormous number of variables and parameters, and becoming important to examine the performance of proposed tensor-based approach.

The remainder of this paper is organized as follow. Notations on tensors and formulation of tensor robust principal component analysis for abnormal traffic pattern extraction are explained in section 2. Continuum modeling approach and the application of proposed methodology to simulated data are described in section 3. The last section is devoted to the conclusion.

2. Tensor-based abnormal traffic pattern extraction

2.1. Notations on tensors

In current subsection mathematical notations on tensors and key definitions used throughout the paper are explained and mainly adopted from Kolda and Bader (2009). Tensor is a multidimensional array, higher order generalization of such mathematical objects as scalars (0^{th}-order tensor), vectors (1^{st}-order tensor) and matrices (2^{nd}-order tensor). An N^{th}-order tensor is denoted with boldface Euler script letter, e.g. $\boldsymbol{\mathcal{X}}$. Vectors and matrices are denoted with boldface lowercase, e.g. $\boldsymbol{x}$, and boldface capital, e.g. $\boldsymbol{X}$, letters respectively. An element of N^{th}-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is a scalar, denoted by $X_{i_1 i_2 \ldots i_N}$. Mathematical operations of addition and subtraction for tensors are defined elementwise, in similar manner as for vectors and matrices. The inner product of two tensors $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is defined as follows

$$\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \rangle = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \ldots \sum_{i_N=1}^{I_N} X_{i_1 i_2 \ldots i_N} Y_{i_1 i_2 \ldots i_N} \tag{1}$$

In order to be able to transform tensor into a matrix representation, the process called matricization or unfolding is introduced as following. Tensor *n*-mode matricization of N^{th}-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is a matrix $X_{[n]}$ formed by mapping from tensor element $(i_1, i_2, \ldots, i_N)$ to a matrix element (i_n, j), with

$$j = 1 + \sum_{\substack{k=1 \\ k \neq n}}^{N} (i_k - 1) J_k \quad and \quad J_k = \prod_{\substack{m=1 \\ m \neq n}}^{k-1} I_m \tag{2}$$

The rank of a tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$, denoted by $rank(\boldsymbol{\mathcal{X}})$, is the smallest positive integer *M*, for which $\boldsymbol{\mathcal{X}}$ could be written as a sum of *M rank-1* tensors (vectors) as follows

$$\boldsymbol{\mathcal{X}} = \sum_{m=1}^{M} r_{m1} \circ r_{m2} \circ \ldots \circ r_{mN} \tag{3}$$

This definition of tensor rank is known to be an NP-hard computational problem, therefore for many applications so-called tensor *n*-rank, which in fact a rank of *n*-mode unfolding, is used

$$rank_n(\boldsymbol{\mathcal{X}}) = rank\left(\boldsymbol{X}_{[n]}\right) \tag{4}$$

In similar way as for matrices, tensor l_p-norm of N^{th}-order tensor $\boldsymbol{\mathcal{X}}$ is introduced and defined as follows

$$\| \boldsymbol{\mathcal{X}} \|_p = \left(\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \ldots \sum_{i_N=1}^{I_N} \left| X_{i_1 i_2 \ldots i_N} \right|^p \right)^{\frac{1}{p}} \tag{5}$$

2.2. Tensor robust principal component analysis

As it was mentioned in introductory part, tensor-based traffic data representation allow to capture underlying multi-mode structure of traffic dynamics and preserve multi-mode correlations. In order to take into account spatiotemporal correlations and similar periodic structure of traffic dynamics (i.e. similar day-to-day dynamics inside a particular urban region), a 3rd order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ with one spatial dimension (I_1) and two temporal dimensions (I_2 and I_3) is constructed. The spatial dimension (I_1) corresponds to a particular location inside urban area. Two temporal dimensions (I_2 and I_3) correspond to time interval within a day and the day respectively. At this point the precise type of traffic data is not specified, however it possibly could be such indicators as traffic speed, flow rate or vehicles density. More importantly that constructed tensor $\boldsymbol{\mathcal{X}}$ contains the information regarding normal and abnormal traffic patterns, which are the subject to be distinguished. In current study it is assumed that normal or expected traffic patterns exhibit similar periodic structure (i.e. day-to-day dynamics inside a particular urban region), so that have so-called low-rank structure. More precisely, this low-rank assumption comes from the fact that day-wise observations of traffic dynamics within particular urban area appear to be similar or correlated to a certain degree. Therefore, these observations are expected to have much fewer degree of freedom than observations of traffic dynamics in an arbitrary regions. Additionally, it is assumed that abnormal or unexpected traffic patterns do not occur very often, and therefore have so-called sparse structure.

Mathematically this could be formulated as following. Given traffic data tensor $\boldsymbol{\mathcal{X}}$ is a subject to be decomposed into low-rank $\boldsymbol{\mathcal{Y}}$ and a sparse $\boldsymbol{\mathcal{Z}}$ components. The sparse tensor $\boldsymbol{\mathcal{Z}}$ contain the information regarding abnormal traffic patterns. The formulation as an optimization problem is as follows

$$\min_{\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}} rank(\boldsymbol{\mathcal{Y}}) + \lambda\|\boldsymbol{\mathcal{Z}}\|_0 \quad s.t. \quad \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Z}} \tag{6}$$

It has been highlighted in previous subsection that estimation the rank of a tensor is known to be an NP-hard computational problem, therefore optimization problem (6) in current formulation is intractable. To overcome this issue appropriate relaxation is needed to be done. To accomplish this, the following procedure, following the style in Xue et al. (2017) is considered. Firstly, the rank of a tensor $rank(\boldsymbol{\mathcal{Y}})$ is substituted by the convex combination of all *n*-ranks, defined by (4). Secondly, all these *n*-ranks and l_0-norm in (6) are replaced with their convex envelopes. These convex envelopes are nuclear $\|\boldsymbol{Y}_{[i]}\|_*$ and l_1-norm for $rank(Y_{[i]})$ and l_0-norm respectively. This leads to the following reformulation of original optimization problem (6) as a robust principal component analysis

$$\min_{\boldsymbol{\mathcal{Y}},\boldsymbol{\mathcal{Z}}} \sum_{i=1}^{3} \alpha_i \|\boldsymbol{Y}_{[i]}\|_* + \lambda\|\boldsymbol{\mathcal{Z}}\|_1 \quad s.t. \quad \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Z}} \tag{7}$$

Need to mention that positive regularization parameter λ in equations (6) and (7) above determines how sparse the component $\boldsymbol{\mathcal{Z}}$ will be. For a larger λ the optimal solution will contain a sparser $\boldsymbol{\mathcal{Z}}$ and a less low-rank $\boldsymbol{\mathcal{Y}}$. On the other hand, a smaller λ will result to a denser $\boldsymbol{\mathcal{Z}}$ and a lower-rank $\boldsymbol{\mathcal{Y}}$. The value of this regularization parameter needs to be tuned to each particular data. The optimization problem (7) is further solved with the help of the method of Augmented Lagrange Multipliers (ALM), described in details in Goldfarb et al. (2014). As a result, two tensors $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$, which are of size as the original tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ are obtained. Tensor $\boldsymbol{\mathcal{Y}}$ contains the information regarding normal or expected traffic patterns. On the other hand, tensor $\boldsymbol{\mathcal{Z}}$ contains the information regarding abnormal traffic patterns. In following section proposed tensor-based approach is validated via applying to simulated data in order to extract abnormal traffic patterns.

3. Continuum modeling of traffic flow

3.1. Model description

One of the major problems with modeling of traffic dynamics in large transportation networks is connected with the fact that existing conventional models such as link-and-node-based traffic flow models could hardly be applicable due to the existence of large number of parameters and variables, as well as significant computational efforts (Sossoe et al., 2015). As one of the possible solutions, recently, so-called continuum modeling approach has been introduced. According to this continuum modeling approach, road network within urban area is assumed to be dense and viewed as a continuum in which travelers can travel in two-dimensional space. This allow to focus on the overall behavior of travelers at the macroscopic level, rather than model and analyze each road link separately. Therefore, less amount of data is required for the model and the problem size could be reduced even in case of large and dense transportation networks (Long et al., 2017).

In current study the model proposed by Long et al. (2017) and Du et al. (2013) is adapted. Firstly, the variables, key definitions and governing equations of original model are explained. Secondly, since the emphasis in current study is placed on abnormal traffic patterns, this continuum-based model of traffic dynamics is supplemented by introduction of so-called disruption field. This disruption field emulates real-world natural or manmade disaster and assumed to be a scalar field with variable parameters, such as intensity and localization.

Mathematically, two-dimensional region Ω with outer boundary $\partial\Omega$ is considered. The travelers are continuously located along $(x, y) \in \Omega$. Following the original model, several variables are introduced

- $\rho(x, y, t)$: density of travellers at location (x, y) at time t with condition on $\rho(x, y, t) = 0 \quad \forall(x, y) \in \partial\Omega$ (travelers are not allowed to leave Ω).
- $q(x, y, t)$: travel demand at location (x, y) at time t.
- $\mathbf{v}(x, y, t) = (v_1(x, y, t), v_2(x, y, t))$: velocity vector at location (x, y) at time t, with $v_1(x, y, t), v_2(x, y, t)$ are X-axis and Y-axis velocity vector components respectively.
- $\mathbf{F}(x, y, t) = (F_1(x, y, t), F_2(x, y, t))$: flow vector at location (x, y) at time t, defined as $\mathbf{F} = \rho\mathbf{v}$, with $F_1(x, y, t), F_2(x, y, t)$ are X-axis and Y-axis flow vector components respectively.
- $c(x, y, t)$: local travel cost per unit distance of travel experienced by travellers.

Regarding the governing equations, the traffic flow is treated as a compressible fluid with following conservation law

$$\rho_t(x, y, t) + \nabla\boldsymbol{F}(x, y, t) = q(x, y, t) \quad \forall(x, y) \in \Omega \;\; \forall t \in T \tag{8}$$

Further, it is assumed that travelers are traveling towards the destination $D \subset \Omega$. In order to describe the decision making process of travelers, following the original model, so-called instantaneous travel cost potential $\varphi(x, y, t)$ is introduced. This cost potential depicts the total travel cost for travelers who depart from location (x, y) at time t to travel to the destination. As in original model, it is assumed that $\varphi(x, y, t) = 0$ at destination and that this cost potential is related to local travel cost as $|\nabla\varphi(x, y, t)| = c(x, y, t)$. Moreover, assuming that travelers choose a path that minimizes their travel cost to the destination, this instantaneous travel cost potential is governed by the equation

$$c(x, y, t)\frac{\boldsymbol{F}(x,y,t)}{|\boldsymbol{F}(x,y,t)|} + \nabla\varphi(x, y, t) = 0 \tag{9}$$

Moreover, in order to emulate the influence of unexpected disruptions, such as natural or manmade disasters, on traffic dynamics, the disruption field is introduced as following. It is assumed that Greenshields's speed-density relationship holds and the aforementioned disruptions lead to capacity reduction. Mathematically, the disruption is assumed to be a scalar field with variable intensity parameter $\gamma(x, y, t) \in [0,1]$, described as follows

$$U(x, y, t) = U_f(x, y)\left(1 - \frac{\rho(x,y,t)}{(1-\alpha_1\gamma(x,y,t))\rho_{jam}(x,y)}\right) \tag{10}$$

Where traffic speed $U(x, y, t) = \|\mathbf{v}(x, y, t)\|$ is equal to the absolute value of velocity vector; $U_f(x, y)$ and $\rho_{jam}(x, y)$ are free-flow speed and jam density at particular location (x, y) respectively. The detailed description of aforementioned continuum model could be found in Du et al. (2013).

3.2. Numerical experiment

Simulation has been conducted in order to validate the proposed tensor-based abnormal traffic pattern extraction method. The simulation setup is the following. We consider a square domain of 1 unit length with travel demand

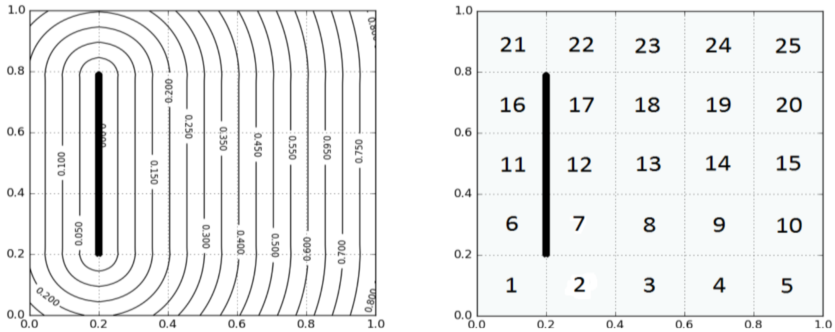

Fig. 1. Simulation domain of 1 square unit length; destination location and the distances to destination (left). Mesh-wise simulation domain separation and numeration of mesh cells (right).

accumulating inside. The travelers are traveling towards the rectangular destination on the left in accordance with continuum-based model described in previous subsection, in particular choosing direction of movement using equation (9). Further, this square domain is divided into 25 uniform mesh cells and the average speed inside each mesh cell is calculated. This simulation setup is depicted on the Fig. 1.

The detailed description and stability analysis of numerical solutions is out of scope of this paper, however, it is worth to mention that simulation domain in Fig.1 was first overlaid by a grid with 0.01 spacing. Further, the solution of conservation law (8) is obtained with the help of two-dimensional extension of Lax-Wendroff scheme, which is second-order accurate in both space and time. Temporal resolution has been chosen in accordance with the Courant–Friedrichs–Lewy (CFL) condition. Additionally, in order to solve boundary value Eikonal-type equation $|\nabla\varphi(x,y,t)| = c(x,y,t)$, fast marching method is utilized. The details of numerical schemes could be found in Warming et al. (1974) and Chopp (2002).

Constructed mesh-wise average speed map was further converted into a tensor-based representation. For this purpose the following procedure was considered. We construct a 3rd order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ with one spatial dimension (I_1) and two temporal dimensions (I_2 and I_3). The spatial dimension (I_1) is of size 25 and corresponds to mesh cells. Two temporal dimensions (I_2 and I_3) are of size (10 and 9) and correspond to number of time intervals within a simulation trial and the number of trials respectively. Data transformation procedure is shown on the Fig.2.

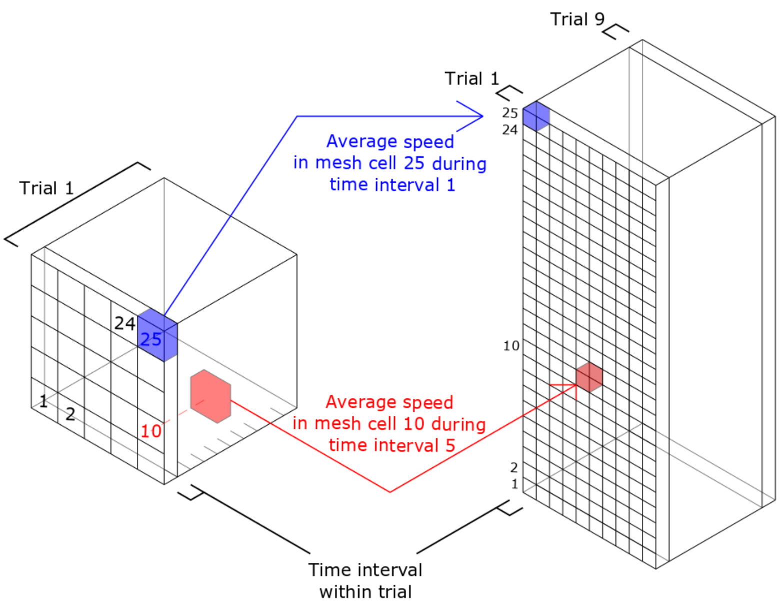

Fig. 2. Mesh-wise average speed map during one trial (left) with numeration according to Fig1. Constructed tensor (right). Colored arrows depict transformation process.

Each of the simulation trials could be thought as evolution of traffic dynamics within one day. In order to investigate the applicability of proposed tensor-based abnormal traffic pattern extraction method, during one of these simulation trials the intensity of disruption field $\gamma(x,y,t)$ was set greater than zero in a randomly chosen sub region inside simulation domain, shown on a Fig.3 (left). Further, tensor robust principal component analysis, described in subsection 2.2, was applied to decompose tensor $\boldsymbol{\mathcal{X}}$ into low-rank $\boldsymbol{\mathcal{Y}}$ and a sparse $\boldsymbol{\mathcal{Z}}$ components.

By analyzing the trial-wise slices of tensor $\boldsymbol{\mathcal{Z}}$ it was further observed that all the slices except the one during which the intensity of disruption field was greater than zero contains zero entries. On the other hand, the slice which corresponds to the trial with anomalous pattern is significantly different. This slice is shown on a Fig.3 (right). From this figure one could observe that the abnormal pattern formed due to the presence of disruption field is noticeable and the mesh cells affected by aforementioned disruption could be correctly identified. The brighter colors on a Fig.3 correspond to the higher values of speed drop (in percentage to the normal conditions). Therefore, the abnormal traffic pattern was correctly identified with the help of proposed tensor-based approach. These results suggests the applicability of proposed method in order to extract abnormal traffic patterns.

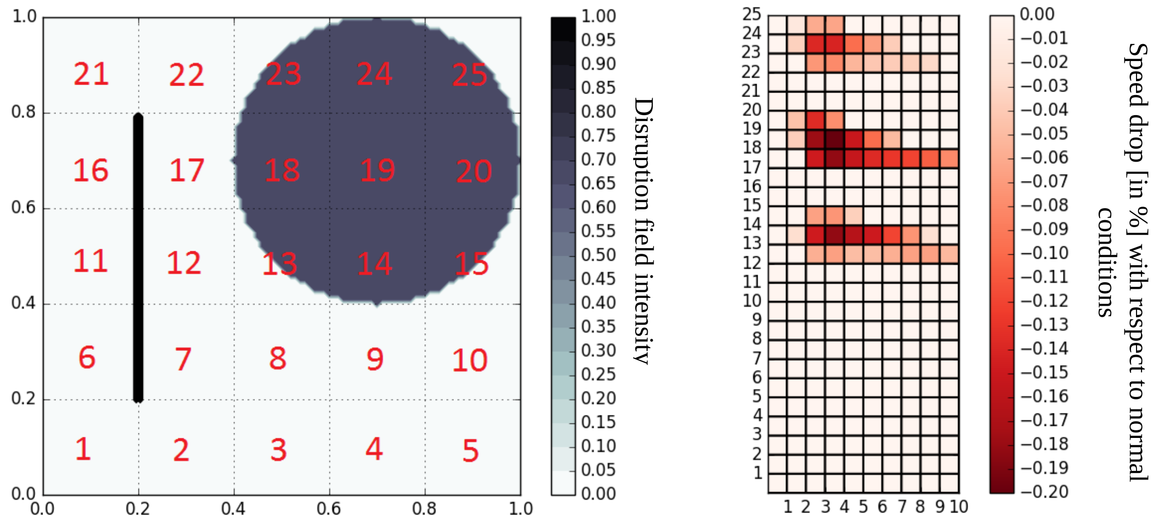

Fig. 3. Disruption field location inside simulation domain and its intensity (left). Corresponding sparse tensor slice (right) with brighter colors correspond to higher values of speed drop, with respect to the normal conditions.

Regarding the possible limitations of proposed tensor-based approach, the following has been observed during the simulation. Motivated by the intention of making simulation scenarios closer to real-life situations, an artificial Gaussian noise had been added in order to emulate observation errors. Under these conditions, the performance of proposed tensor-based approach was slightly declining in case of decreasing of the disruption intensity. This is due to the fact that the influence of disruption to traffic dynamics could be nullified by the fact of presence of aforementioned noise. Nevertheless, more careful analysis is needed to identify the threshold of applicability of proposed approach.

4. Conclusion

This study is devoted to the problem of abnormal traffic pattern extraction in large transportation networks, formed due to the influence of unexpected disruptions, such as natural or manmade disasters. In order to achieve this goal, the following procedure has been proposed. First, in order to take into account complex spatiotemporal structure of traffic dynamics, tensor-based traffic data representation is put forward. Secondly, with the reasonable assumptions on normal or expected traffic dynamics to exhibit similar periodic structure, the problem of abnormal or unexpected traffic patterns detection is treated as a low-rank modeling problem. More precisely, tensor robust principal component analysis has been applied for the purpose of discovering distinctive normal and abnormal traffic patterns. For the validation purposes, continuum modeling approach was employed to emulate large-scale traffic dynamics, with consideration of the effect of aforementioned disruptions. The results suggested the applicability of proposed approach in order to extract abnormal traffic patterns in large transportation networks. However, despite the promising results, additional examination of proposed methodology on real traffic data is necessary and is the subject for further studies.

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